

Radial Symmetry of Large Solutions of Semilinear Elliptic Equations with Convection

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Abstract

We study radial symmetry of large solutions of the semi-linear elliptic problem $\Delta u + \nabla h \cdot \nabla u = f(|x|, u)$, and we provide sharp conditions under which the problem has a radial solution. The result is independent of the rate of growth of the solution at infinity.

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1 Introduction

Radial symmetry of the solutions of $\Delta u = f(|x|, u)$ on \mathbb{R}^n is a well-studied problem and various conditions on the rate of growth and monotonicity of $f(|x|, u)$ as well as behaviour of $u(x)$ at infinity have been presented to guarantee radial symmetry of the solutions. In this paper we study radial symmetry of large solutions of the semi-linear elliptic problem

$$\begin{cases} \Delta u(x) + \nabla h(x) \cdot \nabla u(x) = f(|x|, u(x)) & x \in \mathbb{R}^n \quad (n \geq 2), \\ u(x) \longrightarrow \infty & x \rightarrow \infty. \end{cases} \quad (1)$$

We assume that for large values of $|x|$ and u the function $f(|x|, u)$ is positive and superlinear, and that $\lim_{|x| \rightarrow \infty} u(x) = \infty$ but we do not assume a particular rate of growth at infinity for the solution. Our main focus is the effect of the convection term in radial symmetry of the solutions. The case $h \equiv 0$ with similar setting has been studied in [6] and [7], and in

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contrast to the large boundary condition, symmetry of the small solutions $\lim_{|x| \rightarrow \infty} u(x) = 0$ of the same problem has been studied in [3], [1], [2] and [4].

If $u(x)$ is radial then all of the terms in (1) except perhaps $h(x)$ will be radial which automatically implies radial symmetry of $h(x)$ at least whenever u is not constant. Thus, it is natural to assume that h is radial and whenever clear, we abuse the notation $h(x) = h(|x|)$. We ask for the convection term h to satisfy a particular integrability condition given by

$$\exists R \geq 0 \quad , \quad \int_R^\infty e^{-h(r)} r^{1-n} dr < \infty. \quad (2)$$

This condition is shown to be sharp in the sense that if violated, while all the other conditions hold, there are examples with no radial solution. Having this condition on h , a change of variable is proved to be well-defined which converts the radial solutions of the PDE into the solutions of a corresponding ODE. Then the available ODE theory developed in [6] combined with comparison arguments can be used to prove existence and symmetry of the solutions.

2 Statements and Proofs

To set the appropriate conditions on $f(|x|, u)$ we compare it with a function $g(r, s)$ that satisfies the following conditions:

- (c1) $g(r, s)$ and $g_s(r, s)$ are continuous and positive on $\Omega = \{(r, s) \mid r > r_0, s > s_0\}$ where r_0, s_0 are positive constants.
- (c2) $g(r, s)$ is superlinear in s on Ω in the sense there exist $\lambda > 1$ such that that $g(r, vs) \geq v^\lambda g(r, s)$ for all $v > 1$ and $(r, s) \in \Omega$.
- (c3) $p(r)e^{h(r)}g(r, s)$ is monotone in r on Ω , where the function $p(r)$ is given by

$$p(r) := - \int_r^\infty e^{-h(z)} z^{1-n} dz. \quad (3)$$

The following theorem is the main result of this paper.

Theorem 2.1 *Let $h(r)$ be continuous and satisfy (2). Let $f(r, s)$ and $f_s(r, s)$ be continuous and positive. Assume that there exist a function $g(r, s)$ such that*

$$\lim_{(r,s) \rightarrow (\infty, \infty)} \frac{f(r, s)}{g(r, s)} = 1$$

where $g(r, s)$ satisfies (c1-c3). Assume also that $f(r, s)$ is superlinear in s on Ω . Then

- (i) *All C^2 solutions of Problem (1) are radial.*

(ii) If (1) has a C^2 solution then $\exists R \geq 0$, $\exists \hat{u} > 0$ such that

$$- \int_{|x| > R} p(|x|) e^{h(|x|)} f(x, s) dx < \infty \quad \forall s > \hat{u} \quad (4)$$

where $p(r)$ is given by (3).

(iii) If in addition $f(|x|, u)$ satisfies (c3), Condition (4) is also a sufficient condition for existence of a solution to Problem (1).

In [6] and [7] Taliaferro studies relevance of the conditions on $f(r, s)$ for the problem without the convection term. For example it is shown that superlinearity of $f(r, s)$ is a sharp condition for radial symmetry of the solutions of 2.1. Indeed there are non-radial solutions of the problem 2.1 when this condition fails.

Condition (2) on h is a sharp condition in the sense that if it does not hold, then there are cases with no radial solution to Problem (1). To see this let $h(x) = \beta \log(|x|)$. Notice that Condition (2) holds for $\beta > 2 - n$ and it is violated if $\beta \leq 2 - n$. Consider the critical case when $\beta = 2 - n$ and let $f(r, s) = f(s)$ be a superlinear function. We claim that there is no radial solution to (1). Assume for the contrary that there exists a radial solution $u(x) = u(|x|)$. We have

$$\begin{aligned} f(u) &= \Delta u(x) + \nabla \log(|x|^{(2-n)}) \cdot \nabla u(|x|) \\ &= \Delta u(x) + (2-n) \frac{x}{|x|^2} \cdot \frac{x}{|x|} u'(|x|) \end{aligned}$$

Hence

$$\begin{aligned} f(u) &= \left\{ u''(r) + \frac{n-1}{r} u'(r) \right\} + \frac{2-n}{r} u'(r) \\ &= u''(r) + \frac{1}{r} u'(r) \end{aligned}$$

where $r = |x|$. Now define the radial function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $v(y) := u(|y|)$. Then v is a radial solution of the problem $\Delta v = f(v)$ in \mathbb{R}^2 . This is a contradiction because Osserman showed in [5] that for a superlinear function $f(v)$ the problem $\Delta v(x) = f(v)$ has no large solution in \mathbb{R}^2 . Therefore (1) has no radial solution or it has no solution at all, which both indicate necessity of the condition (2).

Based on the condition (2) on $h(x)$ we can use the following change of variables to transform radial solutions of (1) into the corresponding ODE solutions.

Lemma 2.2 *Let $h(r)$ satisfy (2) and let $f(|x|, u)$ satisfy the conditions of Theorem 2.1. Then $u(x)$ is a radial solution of Equation (1) if and only if $z(t) := u(p^{-1}(t))$ solves*

$$\begin{cases} z''(t) = F(t, z(t)), \\ \lim_{t \rightarrow 0^-} z(t) = \infty, \end{cases} \quad (5)$$

where $p(r)$ is given by (3), and $F(t, z)$ is given by

$$F(t, z) := (p^{-1}(t))^{2n-2} e^{2h(p^{-1}(t))} f(p^{-1}(t), z). \quad (6)$$

Proof. Let $r = |x|$, $t = p(r)$, and $z(t) = u(p^{-1}(t))$. This is a valid change of variable because by definition $p(r)$ is continuous and strictly increasing. We have

$$\begin{aligned} f(r, u(r)) &= [u''(r) + \frac{n-1}{r} u'(r)] + h'(r) u'(r) \\ &= p'(r)^2 z''(p(r)) + (p''(r) + \frac{n-1}{r} p'(r) + h'(r) p'(r)) z'(p(r)) \\ &= p'(r)^2 z''(p(r)). \end{aligned}$$

Therefore

$$\begin{aligned} z''(p(r)) &= \frac{1}{p'(r)^2} f(r, z(p(r))) \\ &= e^{2h(r)} r^{2n-2} f(r, z(p(r))) \\ &= F(p(r), z(p(r))). \end{aligned}$$

Also the boundary condition $\lim_{|x| \rightarrow \infty} u(x) = \infty$ is equivalent to $\lim_{t \rightarrow 0^-} z(t) = \infty$ because $\lim_{r \rightarrow \infty} p^{-1}(r) = 0$. \square

Remark. Note that the definition of $F(t, z)$ implies that for large values of t and z both $F(t, z)$ and $F_z(t, z)$ are continuous and non-negative, and that $F(t, z)$ is superlinear in z . This fact is useful when we study the ODE which corresponds to Equation (1). Lemma 2.2 plays an important role in our arguments. In particular, in the proof of Theorem 2.1 we need to construct two sequences of radial functions for the comparison arguments. The sequences can be constructed by the help of Lemma 2.2 from the ODE counterparts described in Lemma 2.4 as follows. Assuming the conditions of Lemma 2.2 hold, for each $M, m > s_0$ and $r_1 > r_0$ there exists an increasing sequence $\{\rho_k\}_{k=1}^\infty \subseteq (r_1, \infty)$ with $\lim_{k \rightarrow \infty} \rho_k = \infty$, and two sequences of C^2 radial functions $\{u_k(x)\}$ and $\{U_k(x)\}$ that

(i) $U_0(x)$ and $u_0(x), u_1(x), \dots$ are radial solutions of

$$\begin{cases} \Delta u(x) + \nabla h(x) \cdot \nabla u(x) = f(|x|, u(x)) & |x| \geq r_1, \\ u(x) = m & |x| = r_1, \end{cases}$$

(ii) $U_1(x), U_2(x), \dots$ are solutions of

$$\begin{cases} \Delta U_k(x) + \nabla h(x) \cdot \nabla U_k(x) = f(|x|, U_k(x)) & r_1 \leq |x| \leq \rho_k, \\ U_k \longrightarrow \infty & |x| \longrightarrow \rho_k^-, \\ U_k(x) = M & |x| = r_1, \end{cases}$$

(iii) $\lim_{|x| \rightarrow \infty} u_0(x) = \lim_{|x| \rightarrow \infty} U_0(x) = \infty$,

(iv) $u_1(x), u_2(x), \dots$ are all bounded as $|x| \rightarrow \infty$,

(v) For each $|x| > r_1$ we have $\lim_{k \rightarrow \infty} u_k(x) = u_0(x)$, and $\lim_{k \rightarrow \infty} U_k(x) = U_0(x)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1(iii). We start by proving part (iii) where we have additional monotonicity condition (c3) on $f(|x|, u)$. In fact, we prove that, having (c3), Condition (4) is both necessary and sufficient for existence of a solution to (1). This fact will be useful in the proof of other parts. Let $t = p(|x|)$ where $p(r)$ is given by (3). We have

$$-\int_{\Omega} p(|x|) e^{h(|x|)} f(|x|, s) dx = -\sigma_n \int_R^{\infty} r^{n-1} p(r) e^{h(r)} f(r, s) dr,$$

where σ_n is the perimeter of the unit ball in \mathbb{R}^n . Therefore

$$\begin{aligned} -\int_{\Omega} p(|x|) e^{h(|x|)} f(x, s) dx &= -\sigma_n \int_R^{\infty} r^{2n-2} p(r) e^{2h(r)} f(r, s) (e^{-h(r)} r^{(1-n)}) dr \\ &= -\sigma_n \int_R^{\infty} p(r) F(p(r), s) p'(r) dr \\ &= -\sigma_n \int_{t_0}^0 t F(t, s) dt, \end{aligned} \tag{7}$$

where in the second equality we used the fact that $p'(r) = e^{-h(r)} r^{(1-n)}$. Assuming that (4) holds, (7) implies that

$$-\int_{t_0}^0 t F(t, s) dt < \infty \quad \forall s > \hat{u}. \tag{8}$$

By the ODE lemma 2.3 the condition (8) is a necessary and sufficient condition for existence of a solution $z(t)$ of $z''(t) = F(t, z(t))$. By Lemma 2.2 the solution $z(t)$ of the ODE $z''(t) = F(t, z(t))$ can be transformed into a radial solution $u(x) = z(p(|x|))$ of the equation (1). Conversely, if there is a radial solution to (1), using Lemma 2.2, we can transform it into a solution of $z'' = F(t, z)$. This implies that (8) holds. Therefore by (7) we have that Condition (4) is true. \square

Proof of Theorem 2.1(i). Let $g(r, s)$ be as in the statement of the theorem. Because $\lim_{(|x|, s) \rightarrow (\infty, \infty)} \frac{f(x, s)}{g(x, s)} = 1$, without loss of generality we can assume that r_0 and s_0 are large

enough so that $g(r, s) < 2f(r, s)$ on Ω . Define $l(r, s) := \frac{1}{2}g(r, s)$. We work with $l(r, s)$ because we want to use the monotonicity condition (c3) which is not available for $f(|x|, u)$. We claim that if (1) has a solution, then the problem

$$\begin{cases} \Delta y(x) + \nabla h(x) \cdot \nabla y(x) = l(|x|, u(x)) & x \in \mathbb{R}^n \quad (n \geq 2), \\ y(x) \longrightarrow \infty & x \rightarrow \infty. \end{cases} \quad (9)$$

has a radial solution. Assume for the contrary that (1) has a solution while there is no radial solution to (9). To reach to a contradiction, we study another related PDE. Consider constants $s_1 > s_0$ and $r_1 > r_0$ such that

$$\max_{|x|=r_0} u(x) < s_1 < \max_{|x|=r_1} u(x). \quad (10)$$

These constants exist because $\lim_{|x| \rightarrow \infty} u(x) = \infty$. We want to prove that $\exists r_2 > r_1$ such that there exists a radial solution to the following PDE

$$\begin{cases} \Delta v(x) + \nabla h(x) \cdot \nabla v(x) = l(x, v) & r_0 < |x| < r_2 \\ v(x) = s_1 & |x| = r_0 \\ v(x) = s_1 & |x| = r_1 \\ v(x) \longrightarrow \infty & |x| \longrightarrow r_2^-. \end{cases} \quad (11)$$

Setting $t = p(r)$, $z(t) = v(p^{-1}(t))$, and $F(t, z) = (p^{-1}(t))^{2n-2} e^{2a(p^{-1}(t))} l(p^{-1}(t), z)$ the problem of finding r_2 is equivalent to finding $t_2 \in (t_1, 0)$ such that there exists a solution to

$$\begin{cases} z''(t) = F(t, z) \\ z(t_0) = z(t_1) = s_1 \\ \lim_{t \rightarrow t_2^-} z(t) = \infty. \end{cases} \quad (12)$$

Note that because we assumed (9) has no solution, proof of part (iii) implies that

$$\int_{\Omega} -p(|x|) e^{h(|x|)} l(x, s) dx = \infty, \quad (13)$$

which again part (iii) results in

$$-\int_{t_0}^0 t F(t, s) dt = \infty. \quad (14)$$

On the bounded interval $[t_0, t_1]$ with bounded boundary values $z(t_0) = z(t_1) = s_1$, we can use the Green's function of $\frac{-d^2}{dt^2}$ to find a solution to $z'' = F(t, z)$ on this domain. Let $t_2 > t_1$ be the maximal time where $z(t)$ continuously solves $z'' = F(t, z)$. Since $z''(t) = F(t, z) \geq 0$ and $z(t_0) = z(t_1)$, we have that $z'(t) \geq 0$. There are only three possibilities. The first case is when $t_2 = 0$ and $\lim_{t \rightarrow 0^-} z(t) = \infty$. This possibility is ruled out because (14) implies that

(5) has no solution. The second possibility is that $t_2 = 0$ and $\lim_{t \rightarrow 0^-} z(t) < \infty$. In this case by integrating $z'' = F(t, z)$ twice we have $-\int_{t_1}^0 tF(t, z)dt = z(0^-) - z(t_1) + t_1 z'(t_1) < \infty$ which is a contradiction by (14). The only possibility is that $t_2 \in (t_1, 0)$ and $\lim_{t \rightarrow t_2^-} z(t) = \infty$. Therefore we found t_2 with the required conditions. By converting (12) back into the corresponding PDE, there exists $r_2 = p^{-1}(t_2) \in (r_1, \infty)$ such that there is a radial solution to (11). The set $\Sigma = \{x \in (r_0, r_2) | u(x) > v(x)\}$ is an open and non-empty because of the definition of s_1 . Since $f(r, s) \geq h(r, s)$ on $\Sigma \subset \Omega$, we have

$$\Delta(u - v)(x) + \nabla h(x) \cdot \nabla(u - v)(x) = f(x, u) - l(x, v) > 0 \quad \forall x \in \Sigma.$$

But $u(x) - v(x) = 0$ on $\partial\Sigma$. This is a contradiction by the maximum principle. Hence assumption (13) is not true. Therefore if Problem (1) has a solution, then $\int_{\Omega} -p(|x|)e^{h(|x|)}l(x, s)dx < \infty$. Because $\lim_{(r,s) \rightarrow (\infty, \infty)} \frac{f(r,s)}{l(r,s)} = 1/2$ we have

$$\exists \hat{s} \geq s_0 \quad \int_{|x| > r_0} -p(|x|)e^{h(|x|)}f(x, s)dx < \infty \quad \forall s > \hat{s}.$$

□

Proof of Theorem 2.1(ii). We start by showing that the difference of any two C^2 solutions $u_a(x)$ and $u_b(x)$ of PDE (1) goes to zero at infinity. First assume that $y_a(x)$ and $y_b(x)$ are two radial solutions of the PDE. By setting

$$t = p(r), \quad z(t) = y(p^{-1}(t)), \quad F(t, z) := p^{-1}(t)^{2n-2}e^{2h(p^{-1}(t))}f(p^{-1}(t), z),$$

Lemma 2.2 implies that we can find two solutions $z_a(p(|x|)) = y_a(x)$ and $z_b(p(|x|)) = y_b(x)$ of the corresponding ODE. By Lemma 2.3 the difference of any two large solutions of the ODE $z'' = F(t, z)$ goes to zero as $t \rightarrow 0^-$. This implies that $\lim_{|x| \rightarrow \infty} |y_a(x) - y_b(x)| = 0$.

Let $r > 0$ be large enough so that $u(x) > s_0$ for $|x| > r$. Let $m = \min_{|x|=r} u_a(x)$ and $M = \max_{|x|=r} u_a(x)$. Now consider the sequences $u_k(x)$ and $U_k(x)$ described in the remark of Lemma 2.2. By the construction $u_a(x) - u_k(x) > 0$ on $|x| = r$ and $\lim_{|x| \rightarrow \infty} u_a(x) - u_k(x) = \infty$. Since $f_s(r, s) \geq 0$ we have

$$\Delta(u_a(x) - u_k(x)) + \nabla h(x) \cdot \nabla(u_a(x) - u_k(x)) = f(x, u_a(x)) - f(x, u_k(x)) \geq 0.$$

Therefore maximum principle implies $u_a(x) \geq u_k(x)$ for all k and all $|x| > r$. Hence

$$u_a(x) \geq u_0(x) = \lim_{k \rightarrow \infty} u_k(x) \quad \text{for } |x| > r.$$

Similarly $u_a(x) \leq U_k(x)$ for $r < |x| < p_k$. Since $\lim_{k \rightarrow \infty} p_k = \infty$ we have $u_a(x) \leq U_0(x)$ on $|x| > r$. Furthermore $u_0(x)$ and $U_0(x)$ are two radial solutions of the problem (1). By the discussion at the beginning of this step $\lim_{|x| \rightarrow \infty} |U_0(x) - u_0(x)| = 0$. Since $u_0(x) \leq u_a(x) \leq U_0(x)$ we have $\lim_{|x| \rightarrow \infty} |u_a(x) - u_0(x)| = 0$. By the similar argument for $u_b(x)$ we

have $\lim_{|x| \rightarrow \infty} |u_b(x) - u_0(x)| = 0$.

Now assume that R is an orthonormal transformation on \mathbb{R}^n . We have

$$\begin{aligned} [\nabla(h(Rx))] \cdot [\nabla(u(Rx))] &= [\nabla(h(Rx))]^T [\nabla(u(Rx))] \\ &= [(\nabla h)(Rx)] R R^T [(\nabla u)(Rx)] \\ &= \nabla h \cdot \nabla u(R(x)) \end{aligned}$$

Furthermore the Laplace operator is interchangeable with orthonormal operators in the sense that for $u_R(x) = u(R(x))$ we have $\Delta u(R(x)) = \Delta u_R(x)$. Therefore for a given solution $u(x)$ of Problem (1) we have

$$\Delta(u_R - u)(x) + \nabla h \cdot \nabla(u_R - u)(x) = f(x, u_R) - f(x, u).$$

By the argument at the beginning of the proof we know that $\lim_{|x| \rightarrow \infty} |u_R - u| = 0$. Because $f_s(r, s) \geq 0$, the maximum principle implies that $u_R \equiv u$. Therefore $u(x)$ is radial. \square

Appendix

In this appendix we gather the statements of ODE lemmas required for our arguments. See [6] for the proofs of the lemmas.

Lemma 2.3 *Let $\Gamma = \{(t, z) | \hat{t} \leq t < 0, 0 < \hat{z} < z\}$ be given. Assume that $F(t, z)$ and $F_z(t, z)$ are C^0 and non-negative on Γ . Assume also that $F_z(t, z)$ is superlinear in z and $F(t, z)$ is monotone in t on Γ . Then the problem*

$$\begin{cases} z''(t) = F(t, z(t)) \\ \lim_{t \rightarrow 0^-} z(t) = \infty \end{cases} \quad (15)$$

has a C^2 solution if and only if there exists $c \in (\hat{t}, 0)$ such that

$$- \int_c^0 t F(t, z) dt < \infty \quad \forall z > \hat{z}. \quad (16)$$

Furthermore, for any pair of solutions $z_1(t), z_2(t)$ to (15) we have

$$\lim_{t \rightarrow 0^-} |z_2(t) - z_1(t)| = 0.$$

Lemma 2.4 *Let $\Gamma = \{(t, z) | \hat{t} \leq t < 0, 0 < \hat{z} < z\}$ be given. Assume that $F(t, z)$ and $F_z(t, z)$ are C^0 and non-negative on Γ . Assume also that $F(t, z)$ is superlinear in z on Γ . Then for each $\bar{z} > \hat{z}$ and $\bar{t} > \hat{t}$ there exists a sequence $\{\rho_k\}_{k=0}^\infty \subseteq (\bar{t}, 0)$ with $\lim_{k \rightarrow \infty} \rho_k = 0$ and two sequence of C^2 functions $\{z(t)\}_{k=0}^\infty$ and $\{Z(t)\}_{k=0}^\infty$ such that*

(i) $Z_0(t)$ and $z_0(t), z_1(t), \dots$ are solutions of

$$\begin{cases} z''(t) = F(t, z(t)) & t \geq t_1 \\ z(\bar{t}) = \bar{z}. \end{cases}$$

(ii) $\forall k \geq 1$, $Z_k(t)$ is a solution of

$$\begin{cases} Z_k''(t) = F(t, Z_k(t)) & t_1 \leq t < \rho_k \\ \lim_{t \rightarrow \rho_k^-} Z_k(t) = \infty \\ Z_k(\bar{t}) = \bar{z}. \end{cases}$$

(iii) $\lim_{t \rightarrow 0^-} z_0(t) = \lim_{t \rightarrow 0^-} Z_0(t) = \infty$.

(iv) $\forall k \geq 1$, $z_k(t)$ is finite as $t \rightarrow 0^-$.

(v) For each $t \in (\bar{t}, 0)$ we have

$$\lim_{k \rightarrow \infty} z_k(t) = z_0(t) \quad \text{and} \quad \lim_{k \rightarrow \infty} Z_k(t) = Z_0(t).$$

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